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THE UNIFORM HEATING AND BENDING OF A TWO-LAYER PLATE†

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The problem of the uniform heating of a two-layer plate is solved. The transversely isotropic layer considered (a soft plate) is in ideal contact with a rigid isotropic thin elastically deformed layer. The ends of the plate are load-free. A boundary layer of the soft plate (a thin contact layer) is introduced, which enables the boundary conditions on the ends of the plate to be formulated in such a way that the problem has a bounded smooth solution [1]. The two-layer plate, generally speaking, is bounded along the axis perpendicular to the axes directed along the length and thickness of the plate. The resultant force and the resultant moment, applied to the end transverse sections, are equal to zero. The exact solution of the temperature problem is sought using the equations of the theory of elasticity. The plane problem of the bending of a two-layer plate acted upon by a uniformly distributed pressure applied to the side surface of an anisotropic layer is solved by a similar method. The ends of the rigid isotropic layer are clamped.

The temperature problem considered, generally speaking, is not a plane or generalized-plane problem, unlike the asymmetrical mixed temperature problem for a transversely isotropic elastic layer, investigated previously in [1], for which a method was proposed for constructing an exact solution in ordinary Fourier series in one coordinate.

1. HEATING OF A TWO-LAYER PLATE. METHOD OF SOLUTION

The problem of the uniform heating of a two-layer plate is solved. The plate is regarded as a transversely isotropic soft layer, which is in ideal contact with a rigid isotropic thin elastically deformed layer. Using the equations of the theory of elasticity, the stress–strain state of the soft layer of length 2L and thickness H, occupying a region

$$|x'| \le L, \quad 0 \le y' \le H$$

and the stress–strain state of the rigid layer of thickness H_N, occupying the region

$$|x'| \le L, \quad H \le y' \le H + H_N$$

is determined.

The two-layer plate, generally speaking, is bounded along the z axis, perpendicular to the axes directed along the length and thickness of the plate. The length Z_0 of the plate along the z axis is such that $L \ll Z_0$.

We will henceforth use dimensionless Cartesian coordinates x, y, referred to L. Then $y = 0$ is the free side surface of the soft layer being investigated, $x = \pm 1$ are the ends of the plate, $y = h$ is the surface of contact with the rigid layer ($h = H/L$), and $y = h + h_N$ is the free side surface of the rigid layer ($h_N = H_N/L$).

The stresses, strains and displacements of the rigid layer will henceforth be denoted by the superscript (N).

For the deformation of the plate along the z axis we have

$$\varepsilon_z = (C_1 + C_2 y)/E_x, \quad 0 \le y \le h + h_N \tag{1.1}$$

where E_x is the modulus of elasticity of the soft layer along the x axis, and C_1 and C_2 are unknown constants.

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We will assume that ε_x, ε_y, ε_z, $\varepsilon_{xy}/2$ are non-zero components of the strains in the two-layer plate, which depend on x and y. In this case, the following conditions in the displacements

$$\partial_z u_\alpha = \partial_z u_\alpha^{(N)}, \quad \alpha = z, x, y, \quad \partial_x u_z = \partial_x u_z^{(N)}$$

are automatically satisfied on the contact surface of the layers $y = h$.

We will write the relation between the stresses and strains of the soft layer, taking into account relation (1.1):

$$\begin{aligned}
E_0\varepsilon_x &= \sigma_x - \nu_y\sigma_y + E_0\alpha_x T(1+\nu) - (C_1 + C_2 y)\nu(1-\nu^2)^{-1} \\
E_0\varepsilon_y &= \omega\sigma_y - \nu_0\sigma_x + E_0(\alpha_y + k\nu'\alpha_x)T - (C_1 + C_2 y)k\nu'(1-\nu^2)^{-1} \\
E_0\varepsilon_{xy} &= \gamma_0\sigma_{xy}, \quad \sigma_z = \nu\sigma_x + k\nu'\sigma_y - E_x\alpha_x T + C_1 + C_2 y \\
\omega &= \frac{k-(k\nu')^2}{1-\nu^2}, \quad \nu_0 = \frac{k\nu'}{1-\nu}, \quad \gamma_0 = \frac{\gamma}{1-\nu^2}, \quad \gamma = \frac{E_x}{G}, \quad E_0 = \frac{E_x}{1-\nu^2}
\end{aligned} \tag{1.2}$$

Here we have assumed that the isotropy axis of the material is directed along the y axis, E_x and E_y are the moduli of elasticity along the x and y axes, and G is the shear modulus in the (x, y) plane. The coefficient ν characterizes the transverse compression in the isotropy plane (x, z) for stretching in this plane, and ν' represents the transverse compression for stretching in the direction of the y axis [2]. The quantities α_x and α_y are coefficients of thermal expansion along the x and y axes and T is the temperature increment (T = const).

The relation between the stresses and strains of the rigid layer, taking relation (1.1) into account, has the form

$$\begin{aligned}
E_0^{(N)}\varepsilon_x^{(N)} &= \sigma_x^{(N)} - \beta_0\sigma_y^{(N)} + E_0^{(N)}\alpha T(1+\beta) - a(C_1 + C_2 y)\beta(1-\beta^2)^{-1} \\
E_0^{(N)}\varepsilon_y^{(N)} &= \sigma_y^{(N)} - \beta_0\sigma_x^{(N)} + E_0^{(N)}\alpha T(1+\beta) - a(C_1 + C_2 y)\beta(1-\beta^2)^{-1} \\
E_0^{(N)}(1-\beta)\varepsilon_{xy}^{(N)} &= 2\sigma_{xy}^{(N)}, \quad \sigma_z^{(N)} = \beta(\sigma_x^{(N)} + \sigma_y^{(N)}) - E^{(N)}\alpha T + a(C_1 + C_2 y) \\
E_0^{(N)} &= E^{(N)}(1-\beta^2)^{-1}, \quad \beta_0 = \beta(1-\beta)^{-1}, \quad a = E^{(N)}/E_x
\end{aligned} \tag{1.3}$$

where $E^{(N)}$ is Young's modulus, β is Poisson's ratio and α is the coefficient of thermal expansion.

The equations of anisotropic elasticity, taking relations (1.1) and (1.2) into account, can be written in the form

$$\begin{aligned}
&\omega\partial_{xxxx}^4 F + \mu\partial_{xxyy}^4 F + \partial_{yyyy}^4 F = 0 \\
&\sigma_x = \partial_{yy}^2 F, \quad \sigma_y = \partial_{xx}^2 F, \quad \sigma_{xy} = -\partial_{xy}^2 F; \quad \mu = \gamma_0 - 2\nu_0
\end{aligned} \tag{1.4}$$

Correspondingly, the equations of the isotropic theory of elasticity for the rigid layer can be reduced to a biharmonic equation for the potential F (relation (1.4) with $\omega = 1$ and $\mu = 2$).

We will write the boundary conditions on the side surface $y = 0$ of the soft layer as

$$y = 0: \sigma_y = \sigma_{xy} = 0 \tag{1.5}$$

The conditions on the contact surface $y = h$ have the form

$$y = h: \sigma_y = \sigma_y^{(N)}, \quad \sigma_{xy} = \sigma_{xy}^{(N)} \tag{1.6}$$

$$y = h: \varepsilon_x = \varepsilon_x^{(N)}, \quad \partial_x W = \partial_x W^{(N)} \tag{1.7}$$

where W is the dimensionless displacement (with respect to L) along the y axis.

We will write the boundary conditions on the side surface $y = h + h_N$ of the rigid layer as

$$y = h + h_N:\ \sigma_y^{(N)} = \sigma_{xy}^{(N)} = 0 \tag{1.8}$$

There are no loads on the ends of the two-layer plate:

$$x = \pm 1,\quad 0 \le y \le h:\ \sigma_x = \sigma_{xy} = 0 \tag{1.9}$$

$$x = \pm 1,\quad h \le y \le h + h_N:\ \sigma_x^{(N)} = \sigma_{xy}^{(N)} = 0 \tag{1.10}$$

Since the plate, generally speaking, is bounded along the z axis, we have, by the symmetry of the problem ($s = h + h_N$)

$$\int_0^1\int_0^h \sigma_z dydx + \int_0^1\int_h^s \sigma_z^{(N)} dydx = 0 \tag{1.11}$$

$$\int_0^1\int_0^h \sigma_z (y-h) dydx + \int_0^1\int_h^s \sigma_z^{(N)} (y-h) dydx = 0 \tag{1.12}$$

Relation (1.11) is the condition for the resultant force in the section $z = \text{const}$ to be zero, while (1.12) is the condition for the resultant bending moment in this section to be zero.

The solution of problem (1.1)–(1.12), presumably, has a singularity at the points ($x = \pm 1, y = h$) of the composite plate. We will obtain the finite smooth stress-strain state of the composite plate, which is identical with the solution of system (1.1)–(1.12) everywhere with the exception of a certain small neighbourhood of these points.

The problem can be formulated as follows.

We conventionally divide the soft layer into strips S_n

$$\{y_{n-1} \le y \le y_n\},\quad n = 1, 2, \dots, N-1;\quad y_0 = 0,\quad y_{N-1} = h$$

where N-1 is the total number of strips. The strip S_N: $\{h \leqslant y \leqslant h + h_N\}$ of the rigid layer is of thickness h_N.

The plate S_{N-1} is a thin contact layer of small thickness $\delta_{N-1} \ll h$, specified a priori. In this layer it is required to obtain an exact solution of Eq. (1.4), corresponding to the internal mixed temperature problem, i.e. this solution need only satisfy the boundary conditions on the side surfaces $y = h - \delta_{N-1}$ and $y = h$. This layer will be called the boundary layer [1].

In the other layers of the soft layer it is required to satisfy the necessary boundary conditions on the surfaces $y = y_{n-1}$, $y = y_n$ ($n = 1, 2, \dots, N-2$) and the integral boundary conditions on the ends $x = \pm 1$, which correspond to the free boundary. Generally speaking, these layers can have different thicknesses $\delta_n = y_n - y_{n-1}$. The rigid layer and the boundary layer form a two-layer plate, on the ends of which integral conditions are specified of the free-boundary type.

The exact solution of the problem for the soft layer has the form ($0 \leqslant y \leqslant h$, $n = 1, 2, \dots, N-1$)

$$\sigma_x^{(n)} = \sum_{i=2}^{3} (M_{2i}^{(n)} P_{1,2i}^{(n)} + N_{2i}^{(n)} P_{2,2i}^{(n)}) + D_5^{(n)} P_{1,5}^{(n)} + D_3^{(n)} \xi_n + M_2^{(n)} +$$

$$+ \sum \cos(\pi m x) \sum_{i=1}^{2} [R_{2i-1,m}^{(n)} \operatorname{sh}(\alpha_{i,m}\xi_n) + R_{2i,m}^{(n)} \operatorname{ch}(\alpha_{i,m}\xi_n)] \kappa_i^2$$

$$P_{1,6}^{(n)}(x, y) = 6x^2\xi_n^2 - \mu\xi_n^4,\quad P_{2,6}^{(n)} = x^4 - 1 - \omega\xi_n^4$$

$$P_{1,5}^{(n)} = 3x^2\xi_n - \mu\xi_n^3,\quad P_{1,4}^{(n)} = x^2 - 1 - \mu\xi_n^2,\quad P_{2,4}^{(n)} = -\omega\xi_n^2$$

$$\sigma_y^{(n)} = M_6^{(n)}Q_{1,6}^{(n)} + N_6^{(n)}Q_{2,6}^{(n)} + M_4^{(n)}\xi_n^2 + N_4^{(n)}\left(x^2 - \frac{1}{3}\right) + D_5^{(n)}\xi_n^3 - D_1^{(n)}\xi_n - \tag{1.13}$$

$$-\sum \cos(\pi m x)\sum_{i=1}^{2}[R_{2i-1,m}^{(n)}\mathrm{sh}(\alpha_{i,m}\xi_n) + R_{2i,m}^{(n)}\mathrm{ch}(\alpha_{i,m}\xi_n)]$$

$$Q_{1,6}^{(n)}(x,y) = \xi_n^4 - \left(x^4 - \frac{1}{5}\right)\frac{1}{\omega}, \quad Q_{2,6}^{(n)} = 6x^2\xi_n^2 - \frac{\mu}{\omega}\left(x^4 - \frac{1}{5}\right)$$

$$\sigma_{xy}^{(n)} = -4M_6^{(n)}x\xi_n^3 - 4N_6^{(n)}x^3\xi_n - 2M_4^{(n)}x\xi_n - 3D_5^{(n)}x\xi_n^2 + D_1^{(n)}x +$$

$$+\sum \sin(\pi m x)\sum_{i=1}^{2}[R_{2i-1,m}^{(n)}\mathrm{ch}(\alpha_{i,m}\xi_n) + R_{2i,m}^{(n)}\mathrm{sh}(\alpha_{i,m}\xi_n)]\kappa_i$$

where

$$\sigma_\alpha^{(n)} = \sigma_\alpha^{(n)}(x,y), \quad \alpha = x, y, xy$$

$$\alpha_{i,m} = \pi m \kappa_i, \quad \xi_n = y - y_{n-1}, \quad y_0 = 0, \quad y_{N-1} = h, \quad 0 \le \xi_n \le \delta_n$$

$$2\kappa_{1,2}^2 = \mu \pm (\mu^2 - 4\omega)^{1/2}, \quad \kappa_i > 0, \quad i = 1, 2$$

For a material with a pronounced anisotropy we can assume that $\mu > 2\sqrt{\omega}$ [2].

Here and everywhere henceforth, if the limits of summation are not indicated, the summation is carried out from $m = 1$ to $m = \infty$.

The exact solution of the temperature problem for the rigid layer has the form

$$\sigma_x^{(N)} = \sum_{i=2}^{3}(M_{2i}^{(N)}P_{1,2i}^{(N)} + N_{2i}^{(N)}P_{2,2i}^{(N)}) + D_5^{(N)}P_{1,5}^{(N)} + D_3^{(N)}\xi_n + M_2^{(N)} +$$

$$+\sum \cos(\pi m x)\sum_{i=1}^{4}R_{i,m}^{(N)}f_{i,m}(y)$$

$$f_{1,m} = \mathrm{sh}(\pi m \xi_N), \quad f_{2,m} = \mathrm{ch}(\pi m \xi_N), \quad f_{3,m} = [2(\alpha_m)^{-1}f_{1,m} + \xi_N f_{2,m}]/h_N$$

$$f_{4,m} = [2(\alpha_m)^{-1}f_{2,m} + \xi_N f_{1,m}]/h_N$$

$$P_{1,6}^{(N)}(x,y) = 6x^2\xi_N^2 - 2\xi_N^4, \quad P_{2,6}^{(N)} = x^4 - 1 - \xi_N^4$$

$$P_{1,5}^{(N)} = 3x^2\xi_N - 2\xi_N^3, \quad P_{1,4}^{(N)} = x^2 - 1 - 2\xi_N^4, \quad P_{2,4}^{(N)} = -\xi_N^2$$

$$\sigma_y^{(N)} = M_6^{(N)}Q_{1,6}^{(N)} + N_6^{(N)}Q_{2,6}^{(N)} + M_4^{(N)}\xi_N^2 + N_4^{(N)}\left(x^2 - \frac{1}{3}\right) + D_5^{(N)}\xi_N^3 - D_1^{(N)}\xi_N + N_2^{(N)} -$$

$$-\sum \cos(\pi m x)\sum_{i=1}^{4}R_{i,m}^{(N)}\varphi_{i,m}(y) \tag{1.14}$$

$$\varphi_{1,m} = f_{1,m}, \quad \varphi_{2,m} = f_{2,m}, \quad \varphi_{3,m} = \xi_N f_{2,m}/h_N, \quad \varphi_{4,m} = \xi_N f_{1,m}/h_N$$

$$Q_{1,6}^{(N)}(x,y) = \xi_N^4 - \left(x^4 - \frac{1}{5}\right), \quad Q_{2,6}^{(N)} = 6x^2\xi_N^2 - 2\left(x^4 - \frac{1}{5}\right)$$

$$\sigma_{xy}^{(N)} = -4M_6^{(N)}x\xi_N^3 - 4N_6^{(N)}x^3\xi_N - 2M_4^{(N)}x\xi_N - 3D_5^{(N)}x\xi_N^2 + D_1^{(N)}x +$$

$$+\sum \sin(\pi m x)\sum_{i=1}^{4} R_{i,m}^{(N)}\phi_{i,m}(y)$$

$$\phi_{1,m} = f_{2,m},\quad \phi_{2,m} = f_{1,m},\quad \phi_{3,m} = [(\alpha_m)^{-1}f_{2,m} + \xi_N f_{1,m}]/h_N$$

$$\phi_{4,m} = [(\alpha_m)^{-1}f_{1,m} + \xi_N f_{2,m}]/h_N$$

where

$$\alpha_m = \pi m,\quad \xi_N = y - h,\quad 0 \le \xi_N \le h_N$$

$$\sigma_\alpha^{(N)} = \sigma_\alpha^{(N)}(x, y),\quad \alpha = x, y, xy$$

In formulae (1.1), (1.3) and (1.14) $R_{j,m}^{(n)}$ ($j = 1, 2, 3, 4$), $M_{2j}^{(n)}$ ($j = 1, 2, 3$), $N_{2j}^{(n)}$ ($j = 2, 3$), $D_i^{(n)}$ ($i = 1, 3, 5$) and $N_2^{(N)}$, C_1 and C_2 are constants to be determined during the course of the solution.

The conditions for the layers in the soft layer to be matched can be written in the form [1]

$$y = y_{n-1}\colon \sigma_x^{(n)} = \sigma_x^{(n-1)},\quad \sigma_y^{(n)} = \sigma_y^{(n-1)} \tag{1.15}$$

$$y = y_{n-1}\colon \sigma_{xy}^{(n)} = \sigma_{xy}^{(n-1)},\quad \int_0^x \partial_y\sigma_x^{(n)}dx = \int_0^x \partial_y\sigma_x^{(n-1)}dx \tag{1.16}$$

$n = 2, 3, \ldots, N-1$.

The first equation of (1.15) follows from the continuity of the deformation ε_x along y. The second equality of (1.6) denotes that the quantity $\partial_x W$ is continuous along the y coordinate.

The conditions for the soft layer and the rigid layer to be matched can be written in the form

$$y = h\colon E_0\varepsilon_x^{(N-1)} = \frac{1}{a_0}E_0^{(N)}\varepsilon_x^{(N)},\quad \sigma_y^{(N-1)} = \sigma_y^{(N)}$$

$$y = h\colon \sigma_{xy}^{(N-1)} = \sigma_{xy}^{(N)},\quad \int_0^x \partial_y\sigma_x^{(N-1)}dx - [\mu + \nu_0 - (a_0)^{-1}(2 + \beta_0)]\sigma_{xy}^{(N-1)} = \tag{1.17}$$

$$= (a_0)^{-1}\int_0^x \partial_y\sigma_x^{(N)}dx + (\nu - \beta)(1 - \nu^2)^{-1}C_2 x$$

$$a_0(1 - \beta^2) = a(1 - \nu^2) \tag{1.18}$$

The second equality of (1.18) denotes that the quantity $\partial_x W$ is continuous along the y coordinate on the surface $y = h$.

We will rewrite condition (1.5) on the free surface $y = 0$ of the soft layer

$$y = 0\colon \sigma_y^{(1)} = \sigma_{xy}^{(1)} = 0 \tag{1.19}$$

and on the free surface $y = h + h_N$ of the rigid layer

$$y = h + h_N\colon \sigma_y^{(N)} = \sigma_{xy}^{(N)} = 0 \tag{1.20}$$

We will write the integral boundary conditions on the ends $x = \pm 1$ of the soft layer, taking into account the symmetry of the stresses with respect to x

$$\int \sigma_x^{(n)}(1, y)dy = \int \sigma_x^{(n)}(1, y)\xi_n dy = \int \sigma_{xy}^{(n)}(1, y)dy = 0 \tag{1.21}$$

$$n = 1, 2, \ldots, N-2$$

The upper and lower limits of integration are y_n and y_{n-1} respectively.

It follows from expressions (1.13) that the third condition of (1.21) is automatically satisfied.

The integral conditions on the ends $x = \pm 1$ of the rigid layer and the boundary layer have the form

$$\begin{gathered}\int_0^{h_N} \sigma_x^{(N)}(1, \xi_N)d\xi_N + \int_0^{\delta_{N-1}} \sigma_x^{(N-1)}(1, \xi_{N-1})d\xi_{N-1} = 0 \\ \int_0^{h_N} \sigma_x^{(N)}\xi_N d\xi_N + \int_0^{\delta_{N-1}} \sigma_x^{(N-1)}\xi_{N-1}d\xi_{N-1} - \delta_{N-1}\int_0^{\delta_{N-1}} \sigma_x^{(N-1)}d\xi_{N-1} = 0 \\ \int_0^{h_N} \sigma_{xy}^{(N)}d\xi_N + \int_0^{\delta_{N-1}} \sigma_{xy}^{(N-1)}d\xi_{N-1} = 0\end{gathered} \tag{1.22}$$

The third condition of (1.22) is automatically satisfied.

We will write the boundary conditions in the transverse end sections z = const of the two-layer plate

$$\begin{gathered}\sum_{n=1}^{N}\int_0^1\int_{y_{n-1}}^{y_n} \sigma_z^{(n)}dydx = 0 \\ \sum_{n=1}^{N}\int_0^1\int_{y_{n-1}}^{y_n} \sigma_z^{(n)}(y-h)dydx = 0\end{gathered} \tag{1.23}$$

For the boundary layer we will assume in (1.13) that

$$M_6^{(N-1)} = N_6^{(N-1)} = 0 \tag{1.24}$$

We will also require that the following condition is satisfied

$$M_6^{(N)} + 2N_6^{(N)} = 0 \tag{1.25}$$

The last condition denotes that the second condition of (1.7) is satisfied exactly for any finite number L, where L is the number at which the sums of the Fourier series in formulae (1.13) and (1.14) cut off.

The constants written above are found from Eqs (1.15)–(1.25). Condition (1.25) enables the stability of the process of calculating the required constants to be increased considerably.

The existence of particular solutions in the polynomials of Eq. (1.3) enables the convergence of the Fourier series in expressions (1.13) and (1.14) to be improved.

We will briefly indicate the method of setting up the system of algebraic equations for determining the required constants [1].

We will use the following expansions of the functions in Fourier series

$$2x^2 - x^4 = \frac{7}{15} + 48\sum\frac{(-1)^m}{(\pi m)^4}\cos(\pi m x), \quad x^3 - x = 12\sum\frac{(-1)^m}{(\pi m)^3}\sin(\pi m x) \tag{1.26}$$

The functional equations (1.15) and (1.17), the first equation of (1.19) and the first equation of (1.20) are expanded in basis functions x^2, 1 and $\cos(\pi m x)$. This indicates that in these equations, the polynomial $2x^2 - x^4$ in Eq. (1.26) is expanded in a Fourier series. The algebraic expressions with the factors $\cos(\pi m x)$ and x^2, and also the sum of all the constants (the factor of unity) of this functional equation are then equated to zero.

The functional equations (1.16) and (1.18), the second equation of (1.19) and the second equation of (1.20) are expanded in basis functions x and $\sin(\pi m x)$. The expansion of the function $x^3 - x$ in a Fourier series is used here.

Equations (1.24), (1.25), (1.23) and the first two equations from (1.21) and (1.22) close the system of algebraic equations obtained for determining the required constants in solution (1.13), (1.14) and (1.1).

Since the deformation ε_z is a linear function of the y coordinate, this temperature problem is not formally plane or generalized plane.

Results. Calculations of the dimensionless stresses (referred to the quantity $E_0\alpha_x T$), presented in this section, were carried out for $k = 3$, $\gamma = 6$, $\nu = 0.2$, $\nu' = 0.1$, $h = 0.2$, $h_N = 0.02$, $a = 10^4$, $\alpha = 0.5\alpha_x$, $\beta = 0.4$, $N = 6$, $\delta_{N-1} = 0.2h$, $\delta_1 = 0.7\delta$, $\delta_i = \delta$ $(i = 2, 3, 4)$, $\delta = 0.8h/(N - 2.3)$ and $L = 80$, where L is the number at which the Fourier series with respect to the coordinate x are cut off. The thickness δ_1 of the first layer was taken to be less than the thicknesses of the other layers, in order to improve the approximation of the condition $\delta_x(1, y) = 0$ in the region of the free side surface $y = 0$ of the soft layer.

In Fig. 1 we show the distribution of the dimensionless stresses p_x (the continuous curve), p_y (the dash–dot curve) and p_{xy} (the dashed curve) with respect to x in different sections of the soft layer investigated $0 \leqslant y \leqslant h$. Curves 1, 2, 3 and 4 correspond to the sections $y = 0$, $y = 0.584h$, $y = h - \delta_{N-1} = 0.8h$ and $y = h$.

Below we present the results of a calculation of the stresses $p_x^{(N)}$ on the surfaces $y = h$ and $y = s$, $s = h + h_N$

x	0	0.4	0.6	0.8	1
$p_x^{(N)}(x, h)$	144	132	112	69.2	6.5
$p_x^{(N)}(x, s)$	−134	−124	−104	−63.6	−4.9

A comparison of the results of the calculation of the stresses with subscripts (n) and $(n - 1)$ on the matching surfaces of these solutions $y = y_{n-1}$ $(n = 2, 3, 4, 5)$ confirms the high degree of convergence of the Fourier series in solution (1.13) and (1.14). The calculations also show that the solutions are practically identical when $N \geq 5$.

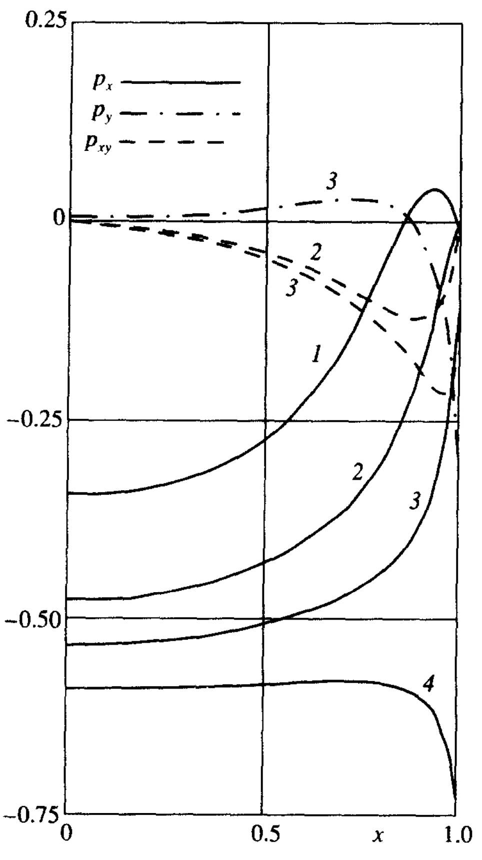

Fig. 1

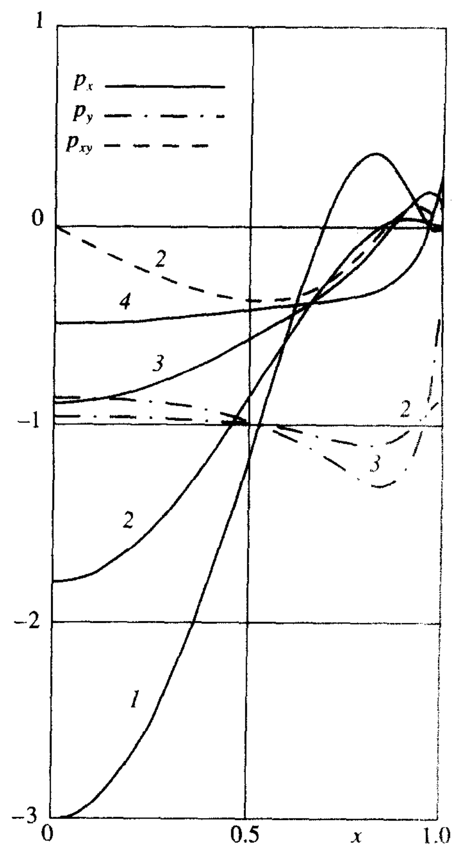

Fig. 2

2. BENDING OF A TWO-LAYER PLATE

The problem of the bending of a two-layer plate was also solved. A transversely isotropic elastic soft layer is in ideal contact with a rigid isotropic thin layer. The ends $x = \pm 1$ of the rigid layer are clamped, and the end of the soft layer are load-free. A uniformly distributed load

$$\sigma_y(x, 0) = -q, \quad q = \text{const.}$$

acts on the free side surface $y = 0$ of the soft layer.

We will use the solution written in Section 1. We have

$$C_1 = C_2 = 0, \quad T = 0, \quad \varepsilon_z = 0 \quad (0 \le y \le h + h_N) \tag{2.1}$$

The stresses in the rigid layer are given by formulae (1.14). The stresses in the soft layer are given by relations (1.13). One must add the term $(-q)$ in formula (1.13) to calculate σ_y.

For the boundary layer S_{N-1} we have condition (1.24)

$$M_6^{(N-1)} = N_6^{(N-1)} = 0$$

It is also necessary to satisfy condition (1.25).

Note that the boundary conditions are not specified directly on the ends of the boundary layer.

Instead of the first condition (1.19) on the side surface $y = 0$ we have

$$y = 0{:}\ \sigma_y^{(1)} = -q$$

Instead of the integral conditions (1.22) on the ends $x = \pm 1$ of the rigid layer we will write the conditions for the end of the rigid plate to be clamped on taking into account the symmetry of the solution of the problem with respect to x

$$x = 1, \quad y = 0.5h_N + h{:}\ u_x^{(N)} = 0, \quad \partial_y u_x^{(N)} = 0; \quad u_x^{(N)} = \int_0^1 \varepsilon_x^{(N)} dx$$

Conditions (1.23) do not arise. The remaining relations for determining the required constants (1.15)–(1.18), (1.20) and (1.21) and the second condition of (1.19) remain without change.

Calculations of the dimensionless stresses (with respect to the quantity q) were carried out for $N = 6$, $\delta_{N-1} = 0.2h$, $\delta_i = \delta$ $(i = 1, 2, 3, 4)$, $\delta = 0.8h/(N-2)$ and $L = 80$. The values of the mechanical constants of the two-layer plate and its geometrical parameters h and h_N are presented in the concluding part of Section 1.

In Fig. 2 we show the distribution of the dimensionless stresses p_x (the continuous curve), p_y (the dash–dot curve) and p_{xy} (the dashed curve) with respect to x in different sections of the soft anisotropic layer investigated $0 \leqslant y \leqslant h$. Curves 1, 2, 3 and 4 correspond to the sections $y = 0$, $y = 0.4h$, $y = h - \delta_{N-1} = 0.8h$, $y = h$.

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